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A new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings

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Abstract

In this paper, we introduce and study a new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings. An existence theorem of solutions is proved without compactness assumptions. A new iterative algorithm is suggested and analysed. The convergence of iterative sequences generated by the new algorithm are also given. As special cases, some known results are also discussed. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, unilateral, obstacle, moving, etc. A useful and important generalization of variational inequalities is a mixed type variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence and algorithm of solutions for the mixed type variational inequalities.

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In 1994, Hassouni and Moudafi [14] used the resolvant operator technique for maximal monotone mapping to study a class of mixed type variational inequalities with single-valued mappings which was called variational inclusions and developed a perturbed algorithm for finding approximate solutions of the mixed variational inequalities. Since then, Adly [1], Huang [15–17], Kazmi [19], Ding [4–8], Ding and Lou [12,13], Noor [22,23], Noor et al. [26], Yuan [37], Uko [33] and Shim et al. [32] have obtained some important extensions and generalizations of the results in [14] in various different directions. We know that one of the most important and difficult problems in variational inequality theory is the development of an efficient and implementable iterative algorithm for solving various classes of variational inequalities and variational inclusions.

In 1989, Chang and Zhu [3] first introduced and studied a class of variational inequalities for fuzzy mappings. Since then, several classes of variational inequalities with fuzzy mappings were considered by Chang and Huang [2], Noor [24], Huang [18], Park and Jeong [28–30], Wu et al. [36] and Ding [9–11].

In 1992 and 1997, by studying an elastoplasticity problem, Panagiotopoulos and Stavroulakis [27] and Noor and Al-Said [25] consider a new class of generalized nonlinear variational inequality problems, which is a variant form and generalization of the problem proposed by Verma [34] and Verma and Base [35].

In this paper, we shall introduce and study a new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings, which includes many new and known classes of generalized mixed variational inequalities and generalized quasivariational inequalities as special cases. An existence theorem of solutions is proved without compactness assumptions. A new iterative algorithm for finding approximate solutions is proposed and analysed. The convergence of the iterative sequences generated by the new algorithm are also given. As special cases, some known results in this field are also discussed.

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{F}(H)$ be a collection of all fuzzy sets over H. A mapping $F: H\to\mathscr{F}(H)$ is said to be a fuzzy mapping. For each $x\in H$, $F(x)$ (denote it by F_x, in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F: H\to\mathscr{F}(H)$ is said to be closed if for each $x\in H$, the function $y\mapsto\mathscr{F}_x(y)$ is upper semicontinuous, i.e., for any given net $\{y_\alpha\}\subset H$ satisfying $y_\alpha\to y_0\in H$, $\limsup_\alpha F_x(y_\alpha) \leqslant F_x(y_0)$. For $A\in\mathscr{F}(H)$ and $\lambda\in[0,1]$, the set $(A)_\lambda=\{x\in H\colon A(x)\geqslant\lambda\}$ is called a λ-cut set of A.

A closed fuzzy mapping $A: H\to\mathscr{F}(H)$ is said to satisfy the condition $(^*)$ if there exists a function $a: H\to[0,1]$ such that for each $x\in H$, $(A_x)_{a(x)}$ is a nonempty bounded subset of H. It is clear that if A is a closed fuzzy mapping satisfying the condition $(^*)$, then for each $x\in H$, the set $(A_x)_{a(x)}\in \mathrm{CB}(H)$, where $\mathrm{CB}(H)$ denotes the family of all nonempty bounded closed subsets of H. In fact, let $\{y_\alpha\}_{\alpha\in\Gamma}\subset(A_x)_{a(x)}$ be a net and $y_\alpha\to y_0\in H$. Then $(A_x)(y_\alpha)\geqslant a(x)$ for each $\alpha\in\Gamma$. Since A is closed, we have

$$A_x(y_0)\geqslant\limsup_{\alpha\in\Gamma}A_x(y_\alpha)\geqslant a(x).$$

This implies $y_0\in(A_x)_{a(x)}$ and so $(A_x)_{a(x)}\in\mathrm{CB}(H)$.

Let $A,B,C,D,E:H\to\mathscr{F}(H)$ be five closed fuzzy mappings satisfying the condition (*). Then there exist five functions $a,b,c,d,e:H\to[0,1]$ such that for each $x\in H$, we have $(A_x)_{a(x)},(B_x)_{b(x)}$, $(C_x)_{c(x)},(D_x)_{d(x)},(E_x)_{e(x)}\in \mathrm{CB}(H)$. Therefore, we can define five set-valued mappings $\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E}$: $H\to\mathrm{CB}(H)$ by

$$\tilde{A}(x)=(A_x)_{a(x)},\quad \tilde{B}(x)=(B_x)_{b(x)},\quad \tilde{C}(x)=(C_x)_{c(x)},\quad \tilde{D}(x)=(D_x)_{d(x)},$$
$$\tilde{E}(x)=(E_x)_{e(x)}$$

for each $x\in H$. In the sequel, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ are called the set-valued mappings induced by the fuzzy mappings A, B, C, D and E, respectively.

Let $N:H\times H\to H$ and $f,g,m:H\to H$ be single-valued mappings and let $A,B,C,D,E:H\to\mathscr{F}(H)$ be fuzzy mappings. Let $a,b,c,d,e:H\to[0,1]$ be given functions. Let $M:H\times H\to 2^H$ be a set-valued mapping such that for each given $z\in H$, $M(\cdot,z):H\to 2^H$ is a maximal monotone mapping with $g(x)-m(y)\in\mathrm{dom}(M(\cdot,z))$ for all $x,y\in H$. Throughout this paper, unless otherwise stated, we will consider the following generalized nonlinear implicit quasivariational inclusion problem with fuzzy mappings:

$$\begin{cases}\text{Find } x,u,v,w,y,z\in H \text{ such that}\\ A_x(u)\geqslant a(x),\quad B_x(v)\geqslant b(x),\quad C_x(w)\geqslant c(x),\quad D_x(z)\geqslant d(x),\\ \quad E_x(y)\geqslant e(x)\quad\text{and}\\ 0\in M(g(x)-m(y),z)+f(w)-N(u,v).\end{cases}\tag{2.1}$$

Special cases:

(I) If $m=0$, $f=g$ and C_x and E_x are both the characteristic function $\chi_{\{x\}}$ of $\{x\}$ and $c(x)=e(x)=1$ for each $x\in H$, then the problem (2.1) reduces to the following problem:

$$\begin{cases}\text{Find } x,u,v,z,\in H \text{ such that}\\ A_x(u)\geqslant a(x),\quad B_x(v)\geqslant b(x),\quad D_x(z)\geqslant d(x)\quad\text{and}\\ 0\in M(g(x),z)+g(x)-N(u,v).\end{cases}\tag{2.2}$$

The problem (2.2) was introduced and studied by Ding [9] which was called generalized implicit quasivariational inclusions with fuzzy mappings. The problems (2.1) and (2.2) include a number of variational inclusions with fuzzy mappings and generalized nonlinear quasivariational inclusions with fuzzy mappings in [2,3,9–11,18,24,28–30,36] as special cases.

(II) If $f=0$, $N=-N$ and C_x is the characteristic function $\chi_{\{x\}}$ of $\{x\}$ and $c(x)=1$ for each $x\in H$, then the problem (2.1) reduces to the following problem:

$$\begin{cases}\text{Find } x,u,v,z,y\in H \text{ such that}\\ A_x(u)\geqslant a(x),\quad B_x(v)\geqslant b(x),\quad D_x(z)\geqslant d(x),\quad E_x(y)\geqslant e(x)\quad\text{and}\\ 0\in M(g(x)-m(y),z)+N(u,v).\end{cases}\tag{2.3}$$

The problem (2.3) is new which is a fuzzy generalization of the generalized set-valued strongly nonlinear quasivariational inclusion considered by Shim et al. [32].

(III) If $A,B,C,D,E:H\to\mathrm{CB}(H)$ are classical set-valued mappings, we can define the fuzzy mappings $A,B,C,D,E:H\to\mathscr{F}(H)$ by

$$x\mapsto\chi_{A(x)},\qquad x\mapsto\chi_{B(x)},\qquad x\mapsto\chi_{C(x)},\qquad x\mapsto\chi_{D(x)},\qquad x\mapsto\chi_{E(x)},$$

where $\chi_{A(x)}$, $\chi_{B(x)}$, $\chi_{C(x)}$, $\chi_{D(x)}$ and $\chi_{E(x)}$ are the characteristic functions of $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$, respectively. Taking $a(x)=b(x)=c(x)=d(x)=e(x)=1$ for all $x\in H$, the problem (2.1) is equivalent to the following problem:

$$\begin{cases} \text{Find } x\in H, \quad u\in A(x), \quad v\in B(x), \quad w\in C(x), \\ \quad z\in D(x) \quad \text{and} \quad y\in E(x) \text{ such that} \\ 0\in M(g(x)-m(y),z)+f(w)-N(u,v). \end{cases} \tag{2.4}$$

For a suitable choice of A, B, C, D, E, f, g, m and M, the problem (2.4) include a number of variational, quasivariational and generalized quasivariational inclusions considered in [1,4,5,14–17,19, 22–24,26,32,33,37] as special cases.

(IV) Let $\phi: H\times H\to \mathbf{R}\cup\{+\infty\}$ be such that for each fixed $z\in H$, $\phi(\cdot,z)$ is a proper convex lower semicontinuous functional satisfying $g(x)-m(y)\in \operatorname{dom}(\partial\phi(\cdot,z))$ for all $x,y\in H$ where $\partial\phi(\cdot,z)$ is the subdifferential of $\phi(\cdot,z)$. By [31], $\partial\phi(\cdot,z): H\to 2^H$ is a maximal monotone mapping. Let $M(x,z)=\partial\phi(x,z)$, $\forall x,z\in H$. For given $z\in H$, by the definition of the subdifferential of $\phi(\cdot,z)$, it is easy to see that the problem (2.1) reduces to the following problem:

$$\begin{cases} \text{Find } x,u,v,w,z,y\in H \text{ such that} \\ A_x(u)\geqslant a(x), \quad B_x(v)\geqslant b(x), \quad C_x(w)\geqslant c(x), \quad D_x(z)\geqslant d(x), \\ \quad E_x(y)\geqslant e(x) \quad \text{and} \\ \langle f(w)-N(u,v),h-g(x)\rangle \geqslant \phi(g(x)-m(y),z)-\phi(h,z), \quad \forall h\in H. \end{cases} \tag{2.5}$$

The problem (2.5) is new and includes many generalized quasivariational inclusion problems considered in [4,14–17,19,23,26,33,37] as special cases.

(V) If $K: H\to 2^H$ is a set-valued mapping such that each $K(x)$ is a closed convex subset of H and for each fixed $z\in H$, $\phi(\cdot,z)=I_{K(z)}(\cdot)$ is the indicator function of $K(z)$,

$$I_{K(z)}(x)=\begin{cases} 0, & \text{if } x\in K(z), \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (2.5) reduces to the generalized strongly nonlinear implicit quasivariational inequality problem with fuzzy mappings:

$$\begin{cases} \text{Find } x,u,v,w,z,y\in H \text{ such that} \\ A_x(u)\geqslant a(x), \quad B_x(v)\geqslant b(x), \quad C_x(w)\geqslant c(x), \quad D_x(z)\geqslant d(x), \\ \quad E_x(y)\geqslant e(x) \quad \text{and} \\ g(x)\in m(y)+K(z), \quad \langle f(w)-N(u,v),h-g(x)\rangle \geqslant 0, \quad \forall h\in m(y)+K(z). \end{cases} \tag{2.6}$$

(VI) If $m(u)=0$ and $N(u,v)=g(v)-u$, $\forall u,v\in H$, D_x and E_x are both the characteristic function $\chi_{\{x\}}$ of $\{x\}$ and $c(x)=e(x)=1$ for each $x\in H$, then the problem (2.6) reduces to the following problem:

$$\begin{cases} \text{Find } x,u,v,w\in H \text{ such that} \\ A_x(u)\geqslant a(x), \quad B_x(v)\geqslant b(x), \quad C_x(w)\geqslant c(x) \quad \text{and} \\ g(x)\in K(x), \quad \langle f(w)-(g(v)-u),h-g(x)\rangle \geqslant 0, \quad \forall h\in K(x). \end{cases} \tag{2.7}$$

The problem (2.7) is called the generalized strongly nonlinear quasivariational inequality problem with fuzzy mappings, which includes the completely generalized strongly quasivariational inequality problems and generalized nonlinear variational inequality problems considered by Park and Jeong [28–30], Noor and Al-Said [25], Verma [34] and Verma and Base [35] as very special cases.

In brief, for appropriate and suitable choices N, A, B, C, D, E, f, g, m, a, b, c, d, e and M, it is easy to see that the generalized nonlinear implicit quasivariational inclusion problem (2.1) includes a number of variational inclusions, generalized quasivariational inclusions, generalized variational inequalities, generalized implicit quasivariational inequalities and generalized implicit quasicomplementarity problems studied by many authors as special cases, for example see [1–5,9,10, 14–19,21–25,28–30,32,33,37] and the references therein. Furthermore, these types of quasivariational inclusions can enable us to study many important nonlinear problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional structural, transportation, elasticity and various applied sciences in a general and unified framework.

Definition 2.1. Let H be a Hilbert space and let $M : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^M : H \to H$ defined by

$$J_\rho^M(x) = (I + \rho M)^{-1}(x), \quad \forall x \in H$$

is said to be the resolvent operator of M, where I is the identity mapping on H.

Lemma 2.1 (Pascali and Sburlan [31]). *Let $M : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^M : H \to H$ of M is nonexpansive, i.e.,*

$$\|J_\rho^M(x) - J_\rho^M(y)\| \leqslant \|x - y\|, \quad \forall x, y \in H.$$

Definition 2.2. A mapping $g : H \to H$ is said to be

(i) δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle g(x) - g(y), x - y\rangle \geqslant \delta\|x - y\|^2, \quad \forall x, y \in H.$$

(ii) σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|g(x) - g(y)\| \leqslant \sigma\|x - y\|, \quad \forall x, y \in H.$$

Definition 2.3. Let $E : H \to 2^H$ and $N : H \times H \to H$ be mappings.

(i) E is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle u_1 - u_2,\ x_1 - x_2\rangle \geqslant \alpha\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in H,\ u_1 \in E(x_1),\ u_2 \in E(x_2),$$

(ii) $N(\cdot,\cdot)$ is said to be α-relaxed Lipschitz with respect to E in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N(u_1,\cdot) - N(u_2,\cdot), x_1 - x_2\rangle \leqslant -\alpha\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in H, u_1 \in E(x_1), u_2 \in E(x_2),$$

(iii) $N(\cdot,\cdot)$ is said to be β-Lipschitz continuous in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \leqslant \beta\|u_1 - u_2\|, \quad \forall u_1, u_2 \in H.$$

In a similar way, we can define the ξ-Lipschitz continuity of $N(\cdot,\cdot)$ in the second argument.

Definition 2.4. A set-valued mapping $E: H \to \mathrm{CB}(H)$ is said to be ε-Lipschitz continuous if there exists a constant $\varepsilon > 0$ such that

$$\tilde{H}(E(x), E(y)) \leqslant \varepsilon\|x - y\|, \quad \forall x, y \in H$$

where $\tilde{H}(\cdot,\cdot)$ is the Hausdorff metric on $\mathrm{CB}(H)$.

3. Iterative algorithm of solutions

We first transfer the problem (2.1) into a fixed point problem.

Theorem 3.1. *(x,u,v,w,z,y) is a solution of the problem (2.1) if and only if (x,u,v,w,z,y) satisfies the following relation*:

$$g(x) = m(y) + J_\rho^{M(\cdot,z)}(g(x) - m(y) - \rho f(w) + \rho N(u,v)), \tag{3.1}$$

where $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$, $z \in \tilde{D}(x)$, $y \in \tilde{E}(x)$ and $\rho > 0$ is a constant.

Proof. By the definition of the resolvent operator $J_\rho^{M(\cdot,z)}$ of $M(\cdot,z)$, we have that (3.1) holds if and only if $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$, $z \in \tilde{D}(x)$ and $y \in \tilde{E}(x)$ such that

$$g(x) - m(y) - \rho f(w) + \rho N(u,v) \in g(x) - m(y) + \rho M(g(x) - m(y), z).$$

The above relations holds if and only if $u \in \tilde{A}(x)$, $v \in \tilde{B}(x) w \in \tilde{C}(x)$, $z \in \tilde{D}(x)$ and $y \in \tilde{E}(x)$ such that

$$0 \in M(g(x) - m(y), z) + f(w) - N(u,v).$$

Hence (x,u,v,w,z,y) is a solution of the problem (2.1) if and only if $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$, $z \in \tilde{D}(x)$ and $y \in \tilde{E}(x)$ are such that (3.1) holds. □

Remark 3.1. Theorem 3.1 is a generalized variant form of Lemma 3.1 of Kazmi [19], Lemma 3.1 of Adly [1], Lemma 2.1 of Huang [15], Lemma 3.1 of Huang [16,17], Theorem 3.1 of Ding [4,5] and Theorem 2.1 of Ding [9]. Eq. (3.1) can be written as

$$x = (1-\lambda)x + \lambda[x - g(x) + m(y) + J_\rho^{M(\cdot,z)}(g(x) - m(y) - \rho f(w) + \rho N(u,v))]. \tag{3.2}$$

This fixed point formulation enables us to suggest the following iterative algorithm.

Algorithm 3.1. Let $A,B,C,D,E: H \to \mathscr{F}(H)$ be closed fuzzy mappings satisfying the condition (*) and $\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E}: \tilde{H} \to \mathrm{CB}(H)$ be the set-valued mappings induced by the fuzzy mappings A, B, C, D and E, respectively. Let $N: H \times H \to H$ and $g,f,m: H \to H$ be single-valued mappings and let

$M: H\times H\to 2^H$ be such that for each fixed $z\in H$, $M(\cdot,z): H\to 2^H$ is a maximal monotone mapping satisfying $g(x)-m(y)\in \mathrm{dom}(M(\cdot,z))$ for all $x,y\in H$. For given $\lambda\in(0,1]$, $x_0\in H$, $u_0\in \tilde{A}(x_0)$, $v_0\in\tilde{B}(x_0)$, $w_0\in\tilde{C}(x_0)$, $z_0\in\tilde{D}(x_0)$ and $y_0\in\tilde{E}(x_0)$, let

$$x_1=(1-\lambda)x_0+\lambda[x_0-g(x_0)+m(y_0)+J_\rho^{M(\cdot,z_0)}(g(x_0)-m(y_0)-\rho f(w_0)+\rho N(u_0,v_0))].$$

By Nadler [20], there exist $u_1\in\tilde{A}(x_1)$, $v_1\in\tilde{B}(x_1)$, $w_1\in\tilde{C}(x_1)$, $z_1\in\tilde{D}(x_1)$, and $y_1\in\tilde{E}(x_1)$ such that

$$\|u_0-u_1\|\leqslant(1+1)\tilde{H}(\tilde{A}(x_0),\tilde{A}(x_1)),$$

$$\|v_0-v_1\|\leqslant(1+1)\tilde{H}(\tilde{B}(x_0),\tilde{B}(x_1)),$$

$$\|w_0-w_1\|\leqslant(1+1)\tilde{H}(\tilde{C}(x_0),\tilde{C}(x_1)),$$

$$\|z_0-z_1\|\leqslant(1+1)\tilde{H}(\tilde{D}(x_0),\tilde{D}(x_1)),$$

$$\|y_0-y_1\|\leqslant(1+1)\tilde{H}(\tilde{E}(x_0),\tilde{E}(x_1)).$$

Let

$$x_2=(1-\lambda)x_1+\lambda[x_1-g(x_1)+m(y_1)+J_\rho^{M(\cdot,z_1)}(g(x_1)-m(y_1)-\rho f(w_1)+\rho N(u_1,v_1))].$$

By induction, we can obtain sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ satisfying

$$\begin{cases} x_{n+1}=(1-\lambda)x_n+\lambda[x_n-g(x_n)+m(y_n)\\ \qquad\quad +J_\rho^{M(\cdot,z_n)}(g(x_n)-m(y_n)-\rho f(w_n)+\rho N(u_n,v_n))],\\ u_n\in\tilde{A}(x_n),\ \|u_n-u_{n+1}\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{A}(x_n),\tilde{A}(x_{n+1})),\\ v_n\in\tilde{B}(x_n),\ \|v_n-v_{n+1}\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{B}(x_n),\tilde{B}(x_{n+1})),\\ w_n\in\tilde{C}(x_n),\ \|w_n-w_{n+1}\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{C}(x_n),\tilde{C}(x_{n+1})),\\ z_n\in\tilde{D}(x_n),\ \|z_n-z_{n+1}\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{D}(x_n),\tilde{D}(x_{n+1})),\\ y_n\in\tilde{E}(x_n),\ \|y_n-y_{n+1}\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{E}(x_n),\tilde{E}(x_{n+1})),\quad n=0,1,\ldots, \end{cases} \tag{3.3}$$

where $\rho>0$ is a constant.

Letting $\lambda=1$ in Algorithm 3.1, we obtain the following algorithm for the problem (2.1) as follows:

Algorithm 3.2. Let $A,B,C,D,E: H\to\mathscr{F}(H)$ be closed fuzzy mappings satisfying the condition $(*)$ and $\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E}: H\to \mathrm{CB}(H)$ be the set-valued mappings induced by the fuzzy mappings, A, B, C, D and E, respectively. Let $N: H\times H\to H$ and $g,f,m: H\to H$ be single-valued mappings and let $M: H\times H\to 2^H$ be such that for each fixed $z\in H$, $M(\cdot,z): H\to 2^H$ is a maximal monotone mapping satisfying $g(x)-m(y)\in\mathrm{dom}(M(\cdot,z))$, for all $x,y\in H$. For given $x_0\in H$, $u_0\in\tilde{A}(x_0)$, $v_0\in\tilde{B}(x_0)$,

$v_0 \in \tilde{B}(x_0)$, $w_0 \in \tilde{C}(x_0)$, $z_0 \in \tilde{D}(x_0)$ and $y_0 \in \tilde{E}(x_0)$, compute $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ from the following iterative schemes:

$$\begin{cases} x_{n+1} = x_n - g(x_n) + m(y_n) + J_\rho^{M(\cdot,z_n)}(g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n)), \\ u_n \in \tilde{A}(x_n),\ \|u_n - u_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(\tilde{A}(x_n), \tilde{A}(x_{n+1})), \\ v_n \in \tilde{B}(x_n),\ \|v_n - v_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(\tilde{B}(x_n), \tilde{B}(x_{n+1})), \\ w_n \in \tilde{C}(x_n),\ \|w_n - w_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(\tilde{C}(x_n), \tilde{C}(x_{n+1})), \\ z_n \in \tilde{D}(x_n),\ \|z_n - z_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(\tilde{D}(x_n), \tilde{D}(x_{n+1})), \\ y_n \in \tilde{E}(x_n),\ \|y_n - y_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(\tilde{E}(x_n), \tilde{E}(x_{n+1})), \quad n = 0, 1, \ldots, \end{cases} \tag{3.4}$$

where $\rho > 0$ is a constant.

From Algorithm 3.1, we can obtain an algorithm for the problem (2.4) as follows:

Algorithm 3.3. Let $A, B, C, D, E : H \to \mathrm{CB}(H)$ be set-valued mappings. Let $N : H \times H \to H$ and $g, f, m : H \to H$ be single-valued mappings and let $M : H \times H \to 2^H$ be such that for each fixed $z \in H$, $M(\cdot, z) : N \to 2^H$ is a maximal monotone mapping satisfying $g(x) - m(y) \in \mathrm{dom}(M(\cdot, z))$ for all $x, y \in H$. For given $x_0 \in H$, $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $w_0 \in C(x_0)$, $z_0 \in D(x_0)$ and $y_0 \in E(x_0)$, compute $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ from the following iterative schemes:

$$\begin{cases} x_{n+1} = (1-\lambda)x_n + \lambda[x_n - g(x_n) + m(y_n) \\ \qquad + J_\rho^{M(\cdot,z_n)}(g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n))], \\ u_n \in A(x_n),\ \|u_n - u_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(A(x_n), A(x_{n+1})), \\ v_n \in B(x_n),\ \|v_n - v_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(B(x_n), B(x_{n+1})), \\ w_n \in C(x_n),\ \|w_n - w_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(C(x_n), C(x_{n+1})), \\ z_n \in D(x_n),\ \|z_n - z_{n+1}\| \leqslant (1 + (1+n)^{-1})\tilde{H}(D(x_n), D(x_{n+1})), \\ y_n \in E(x_n),\ \|y_n - y_{n+1}\| \leqslant (1 + (1+n)^{-1}\tilde{H})(E(x_n), E(x_{n+1})), \quad n = 0, 1, \ldots, \end{cases} \tag{3.5}$$

where $\rho > 0$ is a constant.

From Algorithm 3.1, we can obtain an algorithm for the problem (2.5) as follows:

Algorithm 3.4. Let $A, B, C, D, E : H \to \mathscr{F}(H)$ be closed fuzzy mappings satisfying the condition $(^*)$ and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E} : H \to \mathrm{CB}(H)$ be the set-valued mappings induced by the fuzzy mappings, A, B, C, D and E, respectively. Let $N : H \times H \to H$ and $g, f, m : H \to H$ be single-valued mappings and let $\phi : H \times H \to \mathbf{R} \cup \{+\infty\}$ be such that for each fixed $z \in H$, $\phi(\cdot, z) : H \to \mathbf{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous functional on H satisfying $g(x) - m(y) \in \mathrm{dom}(\partial\phi(\cdot, z))$ for all $x, y \in H$.

For given $x_0 \in H$, $u_0 \in \tilde{A}(x_0)$, $v_0 \in \tilde{B}(x_0)$, $w_0 \in \tilde{C}(x_0)$, $z_0 \in \tilde{D}(x_0)$ and $y_0 \in \tilde{E}(x_0)$, compute $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ from the following iterative schemes:

$$\begin{cases} x_{n+1} = (1-\lambda)x_n + \lambda[x_n - g(x_n) + m(y_n) \\ \qquad\quad + J_\rho^{\partial\phi(\cdot,z_n)}[g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n)], \\ u_n \in \tilde{A}(x_n),\ \|u_n - u_{n+1}\| \leqslant (1+(1+n)^{-1})\tilde{H}(\tilde{A}(x_n), \tilde{A}(x_{n+1})), \\ v_n \in \tilde{B}(x_n),\ \|v_n - v_{n+1}\| \leqslant (1+(1+n)^{-1})\tilde{H}(\tilde{B}(x_n), \tilde{B}(x_{n+1})), \\ w_n \in \tilde{C}(x_n),\ \|w_n - w_{n+1}\| \leqslant (1+(1+n)^{-1})\tilde{H}(\tilde{C}(x_n), \tilde{C}(x_{n+1})), \\ z_n \in \tilde{D}(x_n),\ \|z_n - z_{n+1}\| \leqslant (1+(1+n)^{-1})\tilde{H}(\tilde{D}(x_n), \tilde{D}(x_{n+1})), \\ y_n \in \tilde{E}(x_n),\ \|y_n - y_{n+1}\| \leqslant (1+(1+n)^{-1})\tilde{H}(\tilde{E}(x_n), \tilde{E}(x_{n+1})), \quad n = 0, 1, \ldots, \end{cases} \tag{3.6}$$

where $\rho > 0$ is a constant and $J_\rho^{\partial\phi(\cdot,z)} = (I + \rho\partial\phi(\cdot,z))^{-1}$.

Remark 3.2. In brief, for suitable choices of $A, B, C, D, E, a, b, c, d, e, g, f, m, N$ and M, many iterative algorithms for solving various variational inclusions, generalized quasivariational inclusions, generalized quasivariational inequalities and generalized quasicomplementarity problems in [1–19,21–30, 32–37] can be obtained as special cases of Algorithms 3.1–3.4.

4. Existence and convergence

In this section, we show the existence of a solution of the problem (2.1) and the convergence of the iterative sequences generated by Algorithm 3.1.

Theorem 4.1. *Let $A, B, C, D, E : H \to \mathscr{F}(H)$ be closed fuzzy mappings satisfying the condition $(^*)$ and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E} : H \to \mathrm{CB}(H)$ be the set-valued mappings induced by the fuzzy mappings A, B, C, D, E, respectively. Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ be λ_A -Lipschitz continuous, λ_B -Lipschitz continuous, λ_C-Lipschitz continuous, λ_D-Lipschitz continuous and λ_E-Lipschitz continuous, respectively.* Let *$g : H \to H$ be δ-strongly monotone and σ-Lipschitz continuous and let $N : H \times H \to H$ be α-relaxed Lipschitz with respect to $\tilde{A}$ and β-Lipschitz continuous in the first argument and ξ-Lipschitz continuous in the second argument. Let $m, f : H \to H$ be η-Lipschitz and ε-Lipschitz continuous, respectively. Let $M : H \times H \to 2^H$ be such that for each fixed $z \in H$, $M(\cdot,z) : H \to 2^H$ is a maximal monotone mapping satisfying $g(x) - m(y) \in \mathrm{dom}\, M(\cdot,z)$ for all $x, y \in H$. Suppose that for any $x, y, z \in H$,*

$$\|J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z)\| \leqslant \mu\|x - y\| \tag{4.1}$$

and there exists a constant $\rho > 0$ *such that*

$$\begin{cases} k = 2\sqrt{1-2\delta+\sigma^2} + \mu\lambda_B + \eta\lambda_E, \quad k + \rho(\xi\lambda_B + \varepsilon\lambda_C) < 1, \quad \xi\lambda_B + \varepsilon\lambda_C < \alpha \leqslant \lambda_A\beta, \\ \alpha > (1-k)(\xi\lambda_B + \varepsilon\lambda_C) + \sqrt{(\lambda_A^2\beta^2 - (\xi\lambda_B + \varepsilon\lambda_C)^2)(2k-k^2)}, \\ \left|\rho - \dfrac{\alpha - (1-k)(\xi\lambda_B + \varepsilon\lambda_C)}{\lambda_A^2\beta^2 - (\xi\lambda_B + \varepsilon\lambda_C)^2}\right| \\ \quad < \dfrac{\sqrt{[\alpha - (1-k)(\xi\lambda_B + \varepsilon\lambda_C)]^2 - (\lambda_A^2\beta^2 - (\xi\lambda_B + \varepsilon\lambda_C)^2)(2k-k^2)}}{\lambda_A^2\beta^2 - (\xi\lambda_B + \varepsilon\lambda_C)^2}. \end{cases} \tag{4.2}$$

Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 3.1 *converge strongly to* x^*, u^*, v^*, w^*, z^* *and* y^*, *respectively and* $(x^*, u^*, v^*, w^*, z^*, y^*)$ *is a solution of the generalized nonlinear implicit quasivariational inclusion problem with fuzzy mapping* (2.1).

Proof. By Algorithm 3.1, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\lambda)x_n + \lambda[x_n - g(x_n) + m(y_n) + J_\rho^{M(\cdot,z_n)}(g(x_n) \\ &\quad -m(y_n) - \rho f(w_n) + \rho N(u_n, v_n))] - (1-\lambda)x_{n-1} - \lambda[x_{n-1} - g(x_{n-1}) \\ &\quad + m(y_{n-1}) + J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - m(y_{n-1}) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))]\| \\ &\leqslant (1-\lambda)\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\ &\quad + \lambda\|m(y_n) - m(y_{n-1})\| + \lambda\|J_\rho^{M(\cdot,z_n)}(g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n)) \\ &\quad - J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - m(y_{n-1}) - \rho f(w_n - 1) + \rho(u_{n-1}, v_{n-1}))\|. \end{aligned} \tag{4.3}$$

Since g is δ-strongly monotone and σ-Lipschitz continuous, by using the technique of Noor [21], we have

$$\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \leqslant \sqrt{1-2\delta+\sigma^2}\|x_n - x_{n-1}\|. \tag{4.4}$$

By Lemma 2.1 and condition (4.1), we have

$$\begin{aligned} &\|J_\rho^{M(\cdot,z_n)}(g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n)) \\ &\quad - J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - m(y_{n-1}) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))\| \\ &\leqslant \|J_\rho^{M(\cdot,z_n)}(g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n)) \\ &\quad - J_\rho^{M(\cdot,z_n)}(g(x_{n-1}) - m(y_{n-1}) - \rho f(w_n - 1) + \rho N(u_{n-1}, v_{n-1}))\| \\ &\quad + \|J_\rho^{M(\cdot,z_n)}(g(x_{n-1}) - m(y_{n-1}) - \rho f(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))\| \\ &\quad - J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - m(y_{n-1}) - \rho f(w_n - 1) + \rho N(u_{n-1}, v_{n-1}))\| \\ &\leqslant \|g(x_n) - m(y_n) - \rho f(w_n) + \rho N(u_n, v_n) - g(x_{n-1}) + m(y_{n-1}) + \rho f(w_{n-1}) \\ &\quad - \rho N(u_{n-1}, v_{n-1})\| + \mu\|z_n - z_{n-1}\| \\ &\leqslant \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \|x_n - x_{n-1} + \rho(N(u_n, v_n) - N(u_{n-1}, v_n))\| \end{aligned}$$

$$+\rho\|N(u_{n-1},v_n)-N(u_{n-1},v_{n-1})\|+\|m(y_n)-m(y_{n-1})\|$$
$$+\rho\|f(w_n)-f(w_{n-1})\|+\mu\|z_n-z_{n-1}\|. \tag{4.5}$$

Since $N(\cdot,\cdot)$ is α-relaxed Lipschitz with respect to $\tilde{A}$ and β-Lipschitz continuous in the first argument and $\tilde{A}$ is λ_A-Lipschitz continuous, we have

$$\begin{aligned}
&\|x_n-x_{n-1}+\rho(N(u_n,v_n)-N(u_{n-1},v_n))\|^2\\
&\quad=\|x_n-x_{n-1}\|^2+2\rho\langle N(u_n,v_n)-N(u_{n-1},v_n),x_n-x_{n-1}\rangle\\
&\qquad+\rho^2\|N(u_n,v_n)-N(u_{n-1},v_n)\|^2\\
&\quad\leqslant\|x_n-x_{n-1}\|^2-2\rho\alpha\|x_n-x_{n-1}\|^2+\rho^2\beta^2[(1+n^{-1})\tilde{H}(\tilde{A}(x_n),\tilde{A}(x_{n-1}))]^2\\
&\quad\leqslant(1-2\rho\alpha+\rho^2\beta^2\lambda_A^2(1+n^{-1})^2)\|x_n-x_{n-1}\|^2.
\end{aligned}\tag{4.6}$$

Using ξ-Lipschitz continuity of $N(\cdot,\cdot)$ in the second argument and λ_B-Lipschitz continuity of $\tilde{B}$, we have

$$\begin{aligned}
&\|N(u_{n-1},v_n)-N(u_{n-1},v_{n-1})\|\\
&\quad\leqslant\xi\|v_n-v_{n-1}\|\leqslant\xi(1+n^{-1})\tilde{H}(\tilde{B}(x_n),\tilde{B}(x_{n-1}))\\
&\quad\leqslant\xi\lambda_B(1+n^{-1})\|x_n-x_{n-1}\|.
\end{aligned}\tag{4.7}$$

By the ε-Lipschitz continuity of f and the λ_C-Lipschitz continuity of $\tilde{C}$, we have

$$\begin{aligned}
&\|f(w_n)-f(w_{n-1})\|\leqslant\varepsilon\|w_n-w_{n-1}\|\\
&\quad\leqslant\varepsilon(1+n^{-1})\tilde{H}(\tilde{C}(x_n),\tilde{C}(x_{n-1}))\leqslant\varepsilon\lambda_C(1+n^{-1})\|x_n-x_{n-1}\|.
\end{aligned}\tag{4.8}$$

By the λ_D-Lipschitz continuity of $\tilde{D}$, we have

$$\begin{aligned}
\|z_n-z_{n-1}\|&\leqslant(1+n^{-1})\tilde{H}(\tilde{D}(x_n),\tilde{D}(x_{n-1}))\\
&\leqslant\lambda_D(1+n^{-1})\|x_n-x_{n-1}\|.
\end{aligned}\tag{4.9}$$

By the η-Lipschitz continuity of m and λ_E-Lipschitz continuity of $\tilde{E}$, we have

$$\begin{aligned}
&\|m(y_n)-m(y_{n-1})\|\leqslant\eta\|y_n-y_{n-1}\|\\
&\quad\leqslant\eta(1+n^{-1})\tilde{H}(\tilde{E}(x_n),\tilde{E}(x_{n-1}))\leqslant\eta\lambda_E(1+n^{-1})\|x_n-x_{n-1}\|.
\end{aligned}\tag{4.10}$$

By (4.3)–(4.10), we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\|&\leqslant[(1-\lambda)+2\lambda\sqrt{1-2\delta+\sigma^2}+\lambda\sqrt{1-2\alpha\rho+\lambda_A^2\beta^2\rho^2(1+n^{-1})^2}\\
&\qquad+\lambda(\xi\lambda_B+\varepsilon\lambda_C)\rho(1+n^{-1})+\lambda(\mu\lambda_D+\eta\lambda_E)(1+n^{-1})]\|x_n-x_{n-1}\|\\
&=[\lambda k_n+(1-\lambda)+\lambda t_n(\rho)]\|x_n-x_{n-1}\|\\
&=\theta_n\|x_n-x_{n-1}\|,
\end{aligned}\tag{4.11}$$

where $k_n=2\sqrt{1-2\delta+\sigma^2}+(\mu\lambda_D+\eta\lambda_E)(1+n^{-1})$, $t_n(\rho)=\sqrt{1-2\alpha\rho+\lambda_A^2\beta^2\rho^2(1+n^{-1})^2}+(\xi\lambda_B+\varepsilon\lambda_C)\rho(1+n^{-1})$ and $\theta_n=\lambda k_n+(1-\lambda)+\lambda t_n(\rho)$. Letting $\theta=\lambda k+(1-\lambda)+t(\rho)$ where $k=2\sqrt{1-2\delta+\sigma^2}+$

$\mu\lambda_D+\eta\lambda_E$ and $t(\rho)=\sqrt{1-2\alpha\rho+\lambda_A^2\beta^2\rho^2}+(\xi\lambda_B+\varepsilon\lambda_C)\rho$, we have that $k_n\to k$, $t_n(\rho)\to t(\rho)$ and $\theta_n\to\theta$ as $n\to\infty$. It follows from condition (4.2) that $\theta<1$. Hence $\theta_n<1$ for n sufficiently large. Therefore, (4.11) implies that $\{x_n\}$ is a Cauchy sequence in H and so we can assume that $x_n\to x^*\in H$ as $n\to\infty$. By the Lipschitz continuity of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ we obtain

$$\|u_{n+1}-u_n\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{A}(x_{n+1}),\tilde{A}(x_n))\leqslant(1+(1+n)^{-1})\lambda_A\|x_{n+1}-x_n\|,$$

$$\|v_{n+1}-v_n\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{B}(x_{n+1}),\tilde{B}(x_n))\leqslant(1+(1+n)^{-1})\lambda_B\|x_{n+1}-x_n\|,$$

$$\|w_{n+1}-w_n\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{C}(x_{n+1}),\tilde{C}(x_n))\leqslant(1+(1+n)^{-1})\lambda_C\|x_{n+1}-x_n\|,$$

$$\|z_{n+1}-z_n\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{D}(x_{n+1}),\tilde{D}(x_n))\leqslant(1+(1+n)^{-1})\lambda_D\|x_{n+1}-x_n\|,$$

$$\|y_{n+1}-y_n\|\leqslant(1+(1+n)^{-1})\tilde{H}(\tilde{E}(x_{n+1}),\tilde{E}(x_n))\leqslant(1+(1+n)^{-1})\lambda_E\|x_{n+1}-x_n\|.$$

It follows that $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ are also Cauchy sequences in H. We can assume that $u_n\to u^*$, $v_n\to v^*$, $w_n\to w^*$ $z_n\to z^*$ and $y_n\to y^*$, respectively. Note that $u_n\in\tilde{A}(x_n)$, we have

$$\begin{aligned}d(u^*,\tilde{A}(x^*))&\leqslant\|u^*-u_n\|+d(u_n\tilde{A}(x_n))+\tilde{H}(\tilde{A}(x_n),\tilde{A}(x^*))\\&\leqslant\|u^*-u_n\|+\lambda_A\|x_n-x^*\|\to 0,\quad\text{as } n\to\infty.\end{aligned}$$

Hence we must have $u^*\in\tilde{A}(x^*)$. Similarly, we can show that $v^*\in\tilde{B}(x^*)$, $w^*\in\tilde{C}(x^*)$, $z^*\in\tilde{D}(x^*)$ and $y^*\in\tilde{E}(x^*)$. Hence, we have that $A_{x^*}(u^*)\geqslant a(x^*)$, $B_{x^*}(v^*)\geqslant b(x^*)$, $C_{x^*}(w^*)\geqslant c(x^*)$, $D_{x^*}(z^*)\geqslant d(x^*)$ and $E_{x^*}(y^*)\geqslant e(x^*)$. From $x_{n+1}=(1-\lambda)x_n+\lambda[x_n-g(x_n)+m(y_n)+J_\rho^{M(\cdot,z_n)}(g(x_n)-m(y_n)-\rho f(w_n)+\rho N(u_n,v_n))]$ it follows that

$$g(x^*)=m(y^*)+J_\rho^{M(\cdot,z^*)}(g(x^*)-m(y_*)-\rho f(w^*)+\rho N(u^*,v^*)).$$

By Theorem 3.1, (x^*, u^*, v^*, w^*, z^*, y^*) is a solution of the problem (2.1). □

Remark 4.1. If $\lambda=1$, $f(x)=g(x)$ and $m(x)=0$ for each $x\in H$, then Theorem 4.1 reduces to Theorem 3.1 of Ding [9]. If $m(x)=f(x)=0$ for all $x\in H$, then Theorem 4.1 is a fuzzy extension of Theorem 2.2 of Ding [5].

Theorem 4.2. *Let A, B, C, D, E, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, g, f, m and N be same as in Theorem 4.1. Let $\phi:H\times H\to(-\infty,+\infty]$ be such that for each fixed $z\in H$, $\phi(\cdot,z)$ is a proper convex lower semicontinuous function on H satisfying $g(x)-m(y)\in$ dom $\partial\phi(\cdot,z)$ for all $x,y,z\in H$. Suppose that for all $x,y,z\in H$,*

$$\|J_\rho^{\partial\phi(\cdot,x)}(z)-J_\rho^{\partial\phi(\cdot,y)}(z)\|\leqslant\mu\|x-y\|.$$

If there exist constant $\rho>0$ such that condition (4.2) is satisfied, then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 with $M(x,z)=\partial\phi(x,z)$, $\forall x,z\in H$ strongly converge to x^*, u^*, v^*, w^*, z^* and y^*, respectively and (x^*, u^*, v^*, w^*, z^*, y^*) is a solution *of the problem* (2.5), *that is*

$$\begin{cases} x^*, u^*, v^*, w^*, z^*, y^* \in H \text{ such that} \\ A_{x^*}(u^*) \geqslant a(x^*), \quad B_{x^*}(v^*) \geqslant b(x^*), \quad C_{x^*}(w^*) \geqslant c(x^*), \\ \quad D_{x^*}(z^*) \geqslant d(x^*), \quad E_{x^*}(y^*) \geqslant e(x^*) \quad \text{and} \\ \langle f(w^*) - N(u^*, v^*), \quad u - g(x^*) \rangle \geqslant \phi(g(x^*) - m(y^*), z^*) - \phi(u, z^*), \quad \forall u \in H, \end{cases}$$

Proof. Define a mapping $M : H \times H \to 2^H$ by

$$M(x,z) = \partial\phi(x,z), \quad \forall x, z \in H.$$

Then, it follows from the assumption of ϕ that for each given $z \in H$, $M(\cdot, z) : H \to 2^H$ is a maximal monotone mapping satisfying $g(x) - m(y) \in \operatorname{dom}(M(\cdot, z))$ for all $x, y \in H$. It is easy to see that all conditions of Theorem 4.1 are satisfied. The conclusion of Theorem 4.2 follows from Theorem 4.1. □

Remark 4.2. Theorem 4.2 includes Theorem 3.3 of Ding [9], Theorem 4.1 of Huang [15], Theorem 4.1 of Wu, Long and Huang [36] and the corresponding results of Park and Jeong [28,29] and Noor [24] as very special cases.

Theorem 4.3. *Let $A, B, C, D, E : H \to \mathrm{CB}(H)$ be Lipschitz continuous with Lipschitz constants λ_A, λ_B, λ_C, λ_D, and λ_E, respectively. Let $g, f, m : H \to H$ be such that g is δ-strongly monotone and σ-Lipschitz continuous, f is ε-Lipschitz continuous and m is η-Lipschitz continuous. Let N: $H \times H \to H$ be α-relaxed Lipschitz with respect to A and β-Lipschitz continuous in the first argument and ξ-Lipschitz continuous in the second argument. Let $M : H \times H \to 2^H$ be such that for each fixed $z \in H$, $M(\cdot, z) : H \to 2^H$ is a maximal monotone mapping satisfying $g(x) - m(y) \in \operatorname{dom} M(\cdot, z)$ for all $x, y \in H$. Suppose the conditions* (4.1) *and* (4.2) *in Theorem* 4.1 *hold. Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ generated by Algorithm* 3.3 *converge strongly to x^*, u^*, v^*, w^*, z^* and y^*, respectively and $(x^*, u^*, v^*, w^*, z^*, y^*)$ is a solution of the problem* (2.4).

Proof. Define the fuzzy mappings $A, B, C, D, E : H \to \mathscr{F}(H)$ by

$$x \mapsto \chi_{A(x)} \qquad x \mapsto \chi_{B(x)} \qquad x \mapsto \chi_{C(x)} \qquad x \mapsto \chi_{D(x)} \qquad x \mapsto \chi_{E(x)},$$

where $\chi_{A(x)}$, $\chi_{B(x)}$, $\chi_{C(x)}$, $\chi_{D(x)}$ and $\chi_{E(x)}$ are the characteristic functions of $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$, respectively. Let $a(x) = b(x) = c(x) = d(x) = e(x) = 1$ for all $x \in H$. Then the conclusion of Theorem 4.3 follows from Theorem 4.1. □

Remark 4.3. If $m(x) = f(x) = 0$ for all $x \in H$, then Theorem 4.3 reduces to Theorem 2.2 of Ding [5]. If $f(x) = 0$ and $M(x,z) = M(x)$ for all $x, z \in H$ and $N(u, v)$ is replaced by $-N(u, v)$ for all $u, v \in H$, then Theorem 4.3 reduces to a generalization of Theorems 4.1 and 4.2 of Shim et al. [32] and Theorem 4.1 of Huang [17].

Theorem 4.4. *Let A, B, C, D, E, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, g, m, f and N be the same as in Theorem* 4.1. *Let K: $H \to 2^H$ be such that for each $x \in H$, $K(x)$ is a nonempty closed convex set in H such that the projection operator $P_{K(x)}$ of H onto $K(x)$ satisfies*

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leqslant \mu\|x - y\|, \quad \forall x, y, z \in H.$$

Suppose that there exists a constant $\rho > 0$ such that condition (4.2) *in Theorem* 4.1 *holds. Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{y_n\}$ generated by Algorithm* 3.4 *with $J_\rho^{\partial\phi(\cdot,z)}$ being replaced by $P_{K(z)}$ converge strongly to x^*, u^*, v^*, w^*, z^* and y^*, respectively and $(x^*, u^*, v^*, w^*, z^*, y^*)$ is a solution of the generalized strongly nonlinear implicit quasivariational inequality problem with fuzzy mappings* (2.6).

Proof. For each given $z \in H$, let $\phi(x,z) = I_{K(z)}(x)$ be the indicator function of $K(z)$, i.e.,

$$I_{K(z)}(x) = \begin{cases} 0, & \text{if } x \in K(z), \\ +\infty, & \text{if otherwise.} \end{cases}$$

Then the conclusion follows from Theorem 4.2. □

Remark 4.4. Theorem 4.4 improves and generalizes Theorem 3.4 of Ding [9], Theorem 4.1 of Noor and Al-Said [25] in the following ways: (1) from generalized nonlinear variational inequalities to generalized strongly nonlinear implicit quasivariational inequalities with fuzzy mappings. (2) The mappings $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ may not be compact-valued. Theorem 4.4 also generalizes Theorem 3.1 of Verma [34] and Theorem 3.1 of Verma and Base [35].

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